

Holographic duals to Poisson sigma models

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Poisson sigma models is a very rich class of two-dimensional theories that includes, in particular, all 2D dilaton gravities. By using the Hamiltonian reduction method, we show that a Poisson sigma model (with a sufficiently well-behaving Poisson tensor) on a finite cylinder is equivalent to a noncommutative quantum mechanics for the boundary data.

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I. INTRODUCTION

The holographic principle [1] implies that some quantum theories in $n+1$ dimensions may be fully or partially equivalent to other theories living in n dimensions. One of the manifestations of this principle is the celebrated $\text{AdS}_{n+1}/\text{CFT}_n$ correspondence [2]. One may think that the case $n=1$ is the simplest one. The asymptotic symmetry algebras and their central charges in some AdS_2 gravities were analyzed in, e.g., [3]. A conformal quantum mechanics candidate for the holographic theory was discussed in [4]. However, many conceptual problems remain unsolved. It is not even clear yet whether the dual theory should be a conformal quantum mechanics or a chiral part of a CFT (or, perhaps, both, depending on the model and boundary conditions). The two-dimensional holography definitely deserves more attention, specifically since one may expect some exact results there.

Here we shall consider the Poisson sigma models (PSMs) [5] in 2D that are described by the action

$$S = \int_{\mathcal{M}} d^2\sigma \epsilon^{\mu\nu} \left[X^I \partial_\mu A_{\nu I} + \frac{1}{2} P^{IJ} A_{\mu I} A_{\nu J} \right], \quad (1)$$

where the target space is a Poisson manifold with the coordinates X^I and a Poisson structure $P^{IJ}(X)$, which satisfies the Jacobi identity

$$P^{IL} \partial_L P^{JK} + P^{KL} \partial_L P^{IJ} + P^{JL} \partial_L P^{KI} = 0. \quad (2)$$

$A_{\mu I}$ is a gauge field. $\epsilon^{\mu\nu}$ is the antisymmetric Levi-Civita symbol.

All two-dimensional dilaton gravities [6] are particular cases of PSMs corresponding to a three dimensional target space with $\{A_{\mu I}\} = \{\omega_\mu, e_{\mu a}\}$, where ω_μ is a spin-connection and $e_{\mu a}$ is a zweibein on the 2D space-time. The target space coordinates X^I are the dilaton field X and two auxiliary fields X^a that generate the torsion constraints. It was demonstrated [7], that 2D dilaton gravities are locally quantum trivial, which suggest that all dynamics in that models should reside on the boundary.

An important result was obtained by Cattaneo and Felder [8], who demonstrated that in a PSM on the disc the correlation functions of boundary values X_b^I of X^I can be expressed through the Kontsevich star product [9].

In other words, the dynamics of X_b^I is a noncommutative quantum mechanics obtained by a quantization of the Poisson bracket

$$\{X_b^I, X_b^J\} = P^{IJ}(X_b). \quad (3)$$

Here, we take the world-sheet to be a finite cylinder. To analyze the model (1) we use the Hamiltonian reduction method [10], i.e. before quantization we construct a reduced phase space by solving the constraints and fixing the gauge freedom. This reduced phase space appears to consist of the boundary values of X^I and of the components A_{rI} along the axis of the cylinder. The reduced action contains a vanishing Hamiltonian, but has a non-trivial symplectic structure. Quantization of such an action gives a noncommutative quantum mechanics, see Ref.[11] for some early works.

II. HAMILTONIAN REDUCTION

Let us take the world sheet being a finite cylinder, $\mathcal{M} = S^1 \times [0, l]$. Let t denote a coordinate on S^1 and $r \in [0, l]$. One has to impose some boundary conditions at $r=0, l$ that will ensure the absence of boundary terms in the Euler-Lagrange variation and in the gauge transformation of the action (1). Such conditions are not unique. One possible choice is described below.

The gauge transformations

$$\begin{aligned} \delta_\lambda X^I &= P^{IJ} \lambda_J, \\ \delta_\lambda A_{\mu I} &= -\partial_\mu \lambda_I - \frac{\partial P^{JK}}{\partial X^I} \lambda_K A_{\mu J} \end{aligned} \quad (4)$$

leave the action (1) invariant up to a total derivative,

$$\delta_\lambda S = \int_{\mathcal{M}} d^2\sigma \partial_\mu \left[\epsilon^{\mu\nu} A_{\nu I} \lambda_I \left(P^{IJ} - X^K \frac{\partial P^{IJ}}{\partial X^K} \right) \right]. \quad (5)$$

This boundary term vanishes if we impose the boundary condition

$$A_{tI}|_{\partial\mathcal{M}} = 0. \quad (6)$$

With this boundary condition, the Euler-Lagrange variations of (1) do not produce any boundary terms. The conditions (6) are themselves gauge invariant if

$$\lambda|_{\partial\mathcal{M}} = 0. \quad (7)$$

Our approach is perturbative, but not restricted to a finite order of the perturbation theory. We assume, that the fields are "not too far" from the trivial background $A_{\mu I} = 0 = X^I$ (exactly as in [8]). This will allow us to avoid the difficulties with non-existence of global gauge fixing conditions (Gribov ambiguities). We use the Hamiltonian reduction method [10], that is especially well-suited for first-order theories. Before quantizing, one has to fix the gauge freedom and solve the constraints. The latter are generated by the fields A_{tI} which play the role of Lagrange multipliers. The constraints are

$$\partial_r X^I + P^{IJ} A_{rJ} = 0 \quad (8)$$

On the trivial background $A_{\mu I} = 0 = X^I$, the gauge transformations read: $\delta_\lambda A_{rI} = -\partial_r \lambda_I$. Taking into account the boundary condition (7), one can easily see that the gauge freedom is fixed completely by

$$A_{rI}(x, t) = a_I(t) \quad (9)$$

with some arbitrary functions $a_I(t)$. We shall assume that the condition (9) still selects a representative for each gauge orbit even in a vicinity of the trivial background. Then, if the initial condition $x^I(t) = X^I(t, 0)$ is in a sufficiently small region, the constraint equation (8) has a unique solution $X(r; a_I(t), x^I(t))$. Therefore, the reduced phase space variables are the boundary data $(a_I(t), x^I(t))$, and the action becomes ($\epsilon^{tr} = -1$)

$$S_{\text{red}} = - \int dt Y^I(a(t), x(t)) \partial_t a_I(t), \quad (10)$$

where

$$Y^I(a, x) := \int_0^l dr X^I(r; a, x). \quad (11)$$

The action (10) is 0 + 1 dimensional, it gives a quantum mechanics upon quantization. In terms of the (Y^I, a_I) variables this model is trivial, but in terms of the interesting variables (x^I, a_I) (that are boundary values of the original fields X^I and A_{rI}), the action (10) has a non-trivial symplectic structure, though the Hamiltonian vanishes. The deformation quantization [12] is probably the most appropriate method to quantize such systems. The resulting quantum theory is a noncommutative quantum mechanics with a position-dependent noncommutativity, see [13] for examples of such theories. The constraint equation (8) is non-linear. However, it can be solved perturbatively, which is enough for our purposes.

To derive this result, we have assumed that there is a perhaps small but finite neighborhood of the trivial background such that (9) is an admissible gauge fixing and the constraints have a unique solution for any initial value $x(t)$ from this neighborhood. Then our result is valid to any order of the perturbative expansion around the trivial background (but is not valid non-perturbatively beyond the neighborhood). Not all physically interesting PSMs satisfy this assumption. In some cases, the Poisson

tensor may be rather singular, see Ref. [6]. However, for many reasonable PSMs this procedure does really work, as we shall show below at several examples.

From a somewhat different perspective, the Hamiltonian analysis of PSMs was considered by Strobl [14].

III. PARTICULAR CASES

A. Constant Poisson structure

As a warm-up, let us take

$$P^{IJ} = \frac{2}{l} w^{IJ}, \quad (12)$$

where w^{IJ} is a constant antisymmetric matrix. Then, the gauge transformation (4) of $A_{\mu I}$ is just the gradient transformation, so that the condition (9) is an allowed gauge fixing on the whole phase space. The constraints (8) have a unique solution for arbitrary initial data

$$X^I(r; a, x) = x^I(t) - \frac{2r}{l} w^{IJ} a_J(t) \quad (13)$$

and

$$Y^I(a, x) = l(x^I - w^{IJ} a_J), \quad (14)$$

$$S_{\text{red}} = -l \int dt (x^I - w^{IJ} a_J) \partial_t a_I. \quad (15)$$

The transformation of the canonical coordinates $x^I \rightarrow Y^I/l = x^I - w^{IJ} a_J$ is nothing else than the famous Bopp shift. Upon quantization, this action leads to a noncommutative quantum mechanics, though to a rather trivial one. Since the symplectic structure is constant, the system may be quantized by simply passing to a Moyal type product on the (x^I, a_I) plane.

B. Linear Poisson structure

Let us take a linear Poisson structure,

$$P^{IJ}(X) = C_K^{IJ} X^K. \quad (16)$$

Due to the Jacobi identity on P^{IJ} , the constants C_K^{IJ} have to be structure constants of a Lie algebra. The gauge transformations (4) become just the usual Yang-Mills type transformations, though the gauge group need not be compact in our case. If the gauge group is $SO(2, 1)$ the corresponding PSM is nothing else than the Jackiw-Bunster gravity [15].

The dimension of the space of gauge orbits is locally constant. Therefore, to prove that the gauge condition (9) is admissible, it is enough to check that different functions a^I belong to different gauge orbits. We check an infinitesimal version of this statement. Suppose that a constant a^I and a constant $a^I + \delta a^I$ are related through

a gauge transformation (we omit the t -dependence that is not essential here), i.e.,

$$\partial_r \lambda = -\widehat{C}a\lambda - \delta a, \quad (17)$$

where we suppressed the indices and introduced a matrix $(\widehat{C}a)_I^K \equiv C_I^{JK}a_J$. One can easily find the function $\lambda(r)$ that satisfies (17) and the Dirichlet boundary condition at $r = 0$. It reads

$$\lambda(r) = \exp(-r\widehat{C}a) \int_0^r d\rho \exp(\rho\widehat{C}a) \delta a. \quad (18)$$

Next, we have to check whether there is a choice of δa such that the other boundary condition, $\lambda(l) = 0$, is also satisfied. By a (possibly complex) change of the basis one can bring $\widehat{C}a$ to the canonical Jordan form. Clearly, different Jordan blocks may be considered separately. Let us take one of these blocks,

$$(\widehat{C}a)_1 = \begin{pmatrix} z_0 & z_1 & 0 & \cdots & 0 \\ 0 & z_0 & z_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & z_0 & z_1 \\ 0 & \cdots & 0 & 0 & z_0 \end{pmatrix} \quad (19)$$

with some numbers z_0 and z_1 depending linearly on a_I . An easy computation shows

$$\det \int_0^l d\rho \exp(\rho(\widehat{C}a)_1) = \left[\frac{e^{z_0 l} - 1}{z_0} \right]^n, \quad (20)$$

where n is the dimension of the block. This determinant can be zero only if z_0 is pure imaginary, i.e. if the gauge group has a compact subgroup. Even in this case, for sufficiently small z_0 (equivalently, for sufficiently small a^I) the determinant above is non-zero. Therefore, there is a neighborhood of the trivial vacuum, such that (17) has no solutions, and the gauge condition (9) is admissible.

In the index-free notations the constraint equation (8) reads $\partial_r X - (\widehat{C}a)^T X = 0$, where the transposition means that the index of X is now contracted with the lower index of $\widehat{C}a$ (instead of the upper index in (17)). This equation has a unique solution for any initial data,

$$X(r; a, x) = \exp(r(\widehat{C}a)^T) x, \quad (21)$$

so that the reduced action becomes

$$S_{\text{red}} = - \int dt \left[\int_0^l dr \exp(r(\widehat{C}a)^T) x \right]^I \partial_t a_I. \quad (22)$$

This result admits a geometric interpretation. For example, the solution (21) is a parallel transport of the initial value x^I by a one-parameter group generated by $(\widehat{C}a)^T$.

C. A very short cylinder

Let us assume that the gauge condition (9) is admissible and consider the limit $l \rightarrow 0$. Then, for a

smooth slowly varying Poisson structure $P^{IJ}(X)$, one can approximate $P^{IJ}(X)$ by $P^{IJ}(x)$ in the constraint (8). Hence, $Y^I \simeq l x^I - \frac{1}{2} l^2 P^{IJ}(x) a_J$ and

$$S_{\text{red}} \simeq -l \int dt (x^I - \frac{1}{2} l P^{IJ}(x) a_J) \partial_t a_I. \quad (23)$$

One can easily calculate the Poisson bracket between x^I 's, that reads

$$\{x^I, x^J\} = P^{IJ}(x) + O(l). \quad (24)$$

This reminds us the relation (3), though here it is valid in the limit $l \rightarrow 0$ only.

IV. CONCLUSIONS

We have demonstrated that in the Poisson structure behaves sufficiently well, i.e. if there is a perhaps small but finite vicinity of the trivial vacuum such that the gauge condition (9) is admissible and the constraint (8) has a unique solution, the corresponding Poisson sigma model is (perturbatively) equivalent to a noncommutative quantum mechanics for the boundary data. The example of a linear P^{IJ} shows that regularity of the Poisson structure is not needed.

The present work can be viewed as a extension of the results of [8] from a disc to a finite cylinder. The topology of a cylinder is a natural arena for the Hamiltonian reduction method. Therefore, we were able to extend the noncommutative description to all fields of the model. It would be interesting and important to check our results with other quantization methods, as the ones used in [8].

Our results are not immediately applicable to two-dimensional gravities since the trivial vacuum corresponds to a degenerate metric and is not a natural expansion point. Besides, in gravity models one is usually interested in asymptotic conditions at the conformal boundary of AdS_2 rather than in the conditions at a finite boundary. Although we got a quantum mechanics as a holographic dual, we cannot exclude that for the AdS gravity case the dual theory will be a chiral half of a CFT. The methods suggested here can definitely be extended to AdS gravity models as well.

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